ASYMPTOTICS OF THE FLOWS OF A LIQUID WITH A FREE BOUNDARY WITH VANISHING VISCOSITY

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For the Navier–Stokes equations with vanishing viscosity we consider the plane nonlinear problem of the motion of an incompressible liquid in the region D bounded by the free surface Γ and impenetrable walls S, due to the action of specified initial perturbations:

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v},\nabla)\mathbf{v}=-\nabla p+\varepsilon^2\Delta\mathbf{v}+\mathbf{g},\quad \operatorname{div}\mathbf{v}=0; \tag{1}$$

$$\mathbf{v}|_S=0; \tag{2}$$

$$p-2\varepsilon^2\frac{\partial v_n}{\partial n}=p_*,\quad (x,y)\in\Gamma; \tag{3}$$

$$-4n_xn_y\frac{\partial v_x}{\partial y}+\left(n_x^2-n_y^2\right)\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)=T,\quad (x,y)\in\Gamma; \tag{4}$$

$$\frac{\partial F}{\partial t}+\mathbf{v}\cdot\nabla F=0,\quad (x,y)\in\Gamma; \tag{5}$$

$$\mathbf{v}=\mathbf{v}_*(x,\ y)\ (t=0). \tag{6}$$

All the quantities in Eqs. (1)-(6) are dimensionless. Here $\varepsilon^2 = 1/\mathrm{Re}$ is a small parameter; Re, Reynolds number; $\mathbf{n} = (n_x,n_y)$, unit vector of the internal normal to the free boundary Γ, and $F(x,y,t) = 0$, equation for Γ in implicit form. The liquid is set in motion by an initial velocity field, by initial elevation of the free boundary, and by an external surface stress (p_*, T). The tangential stress on Γ is assumed to be small and of the order of $O(\varepsilon^2)$. The problem is investigated assuming that the solid walls and the free boundary do not have points of contact.

For vanishing viscosity $\varepsilon \to 0$ boundary layers having different properties are formed close to the boundaries of the region. Close to the solid walls there is a layer of infinitely large vorticity of the order of $O(1/\varepsilon)$, and in the neighborhood of the free surface finite vorticity is produced. The equations of the boundary layer are nonlinear in the first case and linear in the second. In the external region (outside the boundary layers) the flow is approximately described by Euler's equations.

The asymptotic expansions of the solution of problem (1)-(6) for low viscosity $\varepsilon \to 0$ take the form

$$\mathbf{v}\sim\sum_{k=0}^{N}\varepsilon^k\mathbf{v}_k(x,y,t)+\sum_{k=0}^{N}\varepsilon^k\mathbf{w}_k(x,y,t;\varepsilon)+\sum_{k=0}^{N}\varepsilon^k\mathbf{h}_k(x,y,t;\varepsilon), \tag{7}$$

$$p\sim\sum_{k=0}^{N}\varepsilon^k p_k(x,y,t)+\sum_{k=0}^{N}\varepsilon^k r_k(x,y,t;\varepsilon)+\sum_{k=0}^{N}\varepsilon^k q_k(x,y,t;\varepsilon),$$

$$\zeta\sim\sum_{k=0}^{N}\varepsilon^k\zeta_k(x,t)$$

($\zeta(x, t)$ is the elevation of the free boundary). We will denote by D_S and D_Γ the regions of the boundary layers close to the solid boundary S and the free surface Γ. Then w_k and r_k are functions of the type of solutions of the problem of the boundary layer in D_S, while $\mathbf{h}_k$ and q_k are functions of the type of solutions of the problem of the boundary layer in D_Γ.

The principal terms of the asymptotics $\mathbf{v}_0$, p_0, and ζ_0 are found from the solution of the problem of the flow of an ideal incompressible liquid in the region D_0 bounded by the wall S and the free boundary Γ_0, due to the action of specified initial perturbation (6)

$$\partial\mathbf{v}_0/\partial t+(\mathbf{v}_0,\ \nabla)\mathbf{v}_0=-\nabla p_0+\mathbf{g},\ \operatorname{div}\mathbf{v}_0=0, \tag{8}$$

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$$\mathbf{v}_0 \cdot \mathbf{n}|_S = 0, \ \mathbf{v}_0 = \mathbf{v}_*, \ \zeta_0 = \zeta_* \ (t = 0).$$

$$p_0 = p_*, \ \partial\zeta_0/\partial t + v_{x0}\partial\zeta_0/\partial x = v_{y0}, \ (x, y) \in \Gamma_0.$$

The functions $\mathbf{v}_k$ and p_k in expansion (7), which define the flow in the region D, are found as a result of the first iterational process [1], and satisfy the linear systems

$$\frac{\partial \mathbf{v}_k}{\partial t} + \sum_{i+j=k} (\mathbf{v}_i, \nabla)\mathbf{v}_j = -\nabla p_k + \Delta \mathbf{v}_{k-2}, \ \operatorname{div} \mathbf{v}_k = 0,$$
$$\mathbf{v}_k|_{t=0} = 0, \ (\mathbf{v}_k + \mathbf{w}_k)\cdot\mathbf{n}|_S = 0, \quad \mathbf{v}_{-1} \equiv 0 \quad (k \geqslant 1), \tag{9}$$

where $\mathbf{n}$ is the vector of the normal to S. The boundary conditions on the free boundary for system (9) will be given below.

The functions of the boundary layer $\mathbf{w}_k$ and r_k manifest themselves in the region D_S in the neighborhood of the boundary S and compensate the discrepancies when the adhesion conditions (2) are satisfied. To simplify the notation we will assume that the boundary S is rectilinear and is described by the equation y = 0. We will determine the equations which satisfy the function $\mathbf{w}_k$ and r_k.

Suppose w_{xk}, w_{yk}, v_{xk}, v_{yk} are the projections of the vectors $\mathbf{w}_k$ and $\mathbf{v}_k$ on the 0_x and 0_y axes, respectively. We substitute the expansions (7) into (1), expand the known coefficients in Taylor series in powers of y, take into account Eqs. (8) and (9), and assume $h_k = q_k = 0$ in D_S apart from terms of a higher order of smallness. We introduce the expansion transformation $y = \varepsilon s$. Equating to zero the coefficients of $\varepsilon^0, \varepsilon^1, \ldots, \varepsilon^N$ in succession, we obtain equations for determining $\mathbf{w}_k$ and r_k. In particular, $h_{y0} = r_0 = r_1 = 0$. Assuming $W_1 = w_{y1} + v_{y1}|_{y=0}$, for w_{x0}, W_1 we derive the following system of nonlinear equations:

$$\frac{\partial w_{x0}}{\partial t} + w_{x0}\frac{\partial w_{x0}}{\partial x} + W_1\frac{\partial w_{x0}}{\partial s} + sa\frac{\partial w_{x0}}{\partial s} + b\frac{\partial w_{x0}}{\partial x} - aw_{x0} = \frac{\partial^2 w_{x0}}{\partial s^2}, \qquad \frac{\partial w_{x0}}{\partial x} + \frac{\partial W_1}{\partial s} = 0 \tag{10}$$

with the following initial and boundary conditions:

$$w_{x0} = W_1 = 0 (t = 0), \ w_{x0} = 0 \ (s = \infty),$$
$$w_{x0} = -b(x, t), \ W_1 = 0 \ (s = 0).$$

Here the coefficients $a(x, t)$, $b(x, t)$ are known if we determine the corresponding flow of an ideal liquid (8)

$$a(x, t) = \partial v_{y0}/\partial y|_{y=0}, \ b(x, t) = v_{x0}|_{y=0}.$$

Note that system (10) leads to the equations of the Prandtl boundary layer [2] if we put

$$u_x = w_{x0} + b, \ u_y = w_{y1} + sa + v_{y1}|_{s=0}.$$

The function r_2 is found from the relation

$$r_2 = \int_{\infty}^{s} \left[\frac{\partial^2 w_{y1}}{\partial s^2} - \frac{\partial w_{y1}}{\partial t} - w_{x0}\frac{\partial w_{y1}}{\partial x} - (W_1 + sa)\frac{\partial w_{y1}}{\partial s} - b\frac{\partial w_{y1}}{\partial s} - aw_{y1} - \left(\frac{\partial v_{y1}}{\partial x}\Big|_{y=0} + s\frac{\partial a}{\partial x}\right)w_{x0}\right] ds.$$

The higher approximations of w_m and r_m satisfy linear equations and are found after solving problem (9) for k = m. We will write the equations to a first approximation

$$\frac{\partial w_{x1}}{\partial t} + W_1\frac{\partial w_{x1}}{\partial s} + w_{x0}\frac{\partial w_{x1}}{\partial x} + sa\frac{\partial w_{x1}}{\partial s} + b\frac{\partial w_{x1}}{\partial x} - aw_{x1} + w_{x1}\frac{\partial w_{x0}}{\partial x} +$$
$$+ W_2\frac{\partial w_{x0}}{\partial s} = \frac{\partial^2 w_{x1}}{\partial s^2} - \left[v_{x1} + s\frac{\partial v_{x0}}{\partial y}\right]_{y=0}\frac{\partial w_{x0}}{\partial x} - \left[\frac{\partial v_{x1}}{\partial x} + s\frac{\partial^2 v_{x0}}{\partial x \partial y}\right]_{y=0} w_{x0} -$$
$$-\frac{\partial v_{x0}}{\partial y}\Big|_{y=0} w_{y1} - \left[s\frac{\partial v_{y1}}{\partial y} + \frac{s^2}{2}\frac{\partial^2 v_{y0}}{\partial y^2}\right]_{y=0}\frac{\partial w_{x0}}{\partial s}, \quad \frac{\partial w_{x1}}{\partial x} + \frac{\partial W_2}{\partial s} = 0$$

with the corresponding boundary and initial conditions

$$w_{x1} = 0 \ (s = \infty), \ w_{x1} + v_{x1} = 0, \ W_2 = 0 \ (s = 0),$$
$$w_{x1} = W_2 = 0 \ (t = 0),$$

where $W_2 = w_{y2} + v_{y2}|_{y=0}$.

We will now determine the boundary-layer functions $\mathbf{h}_k$ and q_k which are concentrated in the neighborhood of the free boundary and compensate the discrepancies which arise when the dynamic condition is satisfied for the tangential stress on Γ. The quantities $\mathbf{h}_k$ and q_k are constructed as in [3]. In the region of Γ we introduce fixed local coordinates ρ, φ given by

$$x = X(t, \varphi) + \rho n_{x0}, \ y = Y(t, \varphi) + \rho n_{y0},$$

where ρ is the distance of the point (x, y) from Γ_0 — the free surface of nonviscous flow (8); $n_{x0}(t, \varphi)$, $n_{y0}(t, \varphi)$, components of the unit vector of the normal to Γ_0, and X(t, φ), y = Y(t, φ), parametric equation of the contour Γ_0. We substitute (7) into (1) and we write the equations obtained in local coordinates. We expand the known coefficients in Taylor series in powers of ρ, we take into account the correctness of the relation $\partial\rho/\partial t + \mathbf{v}_0 \cdot \nabla\rho = 0$ for ρ = 0, and we assume ρ = εξ. Apart from terms of higher order of smallness we assume $w_k = r_k = 0$ in D_Γ. Equating to zero the coefficients of ε^k, for $\mathbf{h}_k$ ($k \geqslant 1$) we obtain the following system of linear equations:

$$\frac{\partial h_{\varphi k}}{\partial t} + \xi a_1(t, \varphi)\frac{\partial h_{\varphi k}}{\partial \xi} + b_1(t, \varphi)\frac{\partial h_{\varphi k}}{\partial \varphi} - a_1 h_{\varphi k} = \frac{\partial^2 h_{\varphi k}}{\partial \xi^2} + F_k, \tag{11}$$
$$\frac{\partial h_{\rho k}}{\partial \xi} - \varkappa \frac{\partial}{\partial \xi}(\xi h_{\rho, k-1}) + \delta^{-1}\frac{\partial h_{\varphi, k-1}}{\partial \varphi} = 0,$$
$$\frac{\partial q_k}{\partial \xi} = -2\left[\varkappa v_{\varphi 0} + \delta^{-1}\frac{\partial v_{\rho 0}}{\partial \varphi}\right]_{\rho=0} h_{\varphi, k-1} + D_k$$

with the boundary conditions

$$\frac{\partial h_{\varphi k}}{\partial \xi} = -\left[\delta^{-1}\frac{\partial v_{\rho, k-1}}{\partial \varphi} + \frac{\partial v_{\varphi, k-1}}{\partial \rho} + \varkappa v_{\varphi, k-1}\right]_{\rho=0} + M_k \ \ (\xi = 0),$$
$$\mathbf{h}_k = q_k = 0 \ \ (\xi = \infty), \ \ \mathbf{h}_k = 0 \ \ (t = 0).$$

The coefficients F_k, D_k, and M_k are known and are not written in view of their complexity, while $F_1 = D_1 = M_1 = D_2 = 0$. Here

$$a_1(t, \varphi) = \frac{\partial}{\partial \rho}[\rho_t + \mathbf{v}_0 \cdot \nabla\rho]_{\rho=0}, \quad b_1(t, \varphi) = [\varphi_t + \mathbf{v}_0 \cdot \nabla\varphi]_{\rho=0},$$

and ϰ and δ are the curvature and Lamé coefficient of the contour Γ_0. Note [3] that $\mathbf{h}_0 = h_{\rho 1} = q_0 = q_1 = 0$. As in [3], the solution of system (11) is found in quadratures.

The functions ζ_k, determining the asymptotic form of the free boundary, are found together with v_k when solving problem (9). The boundary conditions on the boundary Γ for system (9) are obtained by applying the first and second iterational processes [1] simultaneously to condition (3), and in local coordinates have the form

$$p_k + r_k + \zeta_k \frac{\partial p_0}{\partial \rho} - 2\frac{\partial v_{\rho, k-2}}{\partial \rho} + Q_k = 0 \quad (\rho = 0). \tag{12}$$

where $k \geqslant 1$; $\mathbf{v}_{-1} = 0$; $Q_1 = 0$.

Assuming that $F = -\rho + \zeta(t, \varphi)$ in (5) and using the same reasoning as when deriving (12), we obtain the following equations for determining ζ_k:

$$\frac{\partial \zeta_k}{\partial t} + b_1(t, \varphi)\frac{\partial \zeta_k}{\partial \varphi} - a_1(t, \varphi)\zeta_k = [v_{\rho k} + h_{\rho k}]_{\rho=0} + E_k, \tag{13}$$
$$\zeta_k = 0 (t = 0), \ E_1 = 0.$$

Note that $\zeta_0 = 0$ since ρ = 0 is the equation of the boundary Γ_0.

We will now write the problem of determining the external flow in D to a first approximation. The functions v_1 and p_1 together with ζ_1 satisfy system (9) and (13) for k = 1. The boundary condition for y = 0 is obtained by assuming $s = \infty$ in the relation $W_1 = w_{y1} + v_{y1}|_{y=0}$ and assuming that w_{y1} is a function of the boundary-layer type, i.e., $w_{y1} = 0$ ($s = \infty$). Thus, $v_{y1}|_{y=0} = W_1|_{s=\infty}$. The boundary condition on Γ follows from (12) for k = 1. We will assume that the flow of an ideal liquid (8) is potential, i.e., $v_0 = \nabla\Phi_0$, in which case the vector v_1 is also potential ($v_1 = \nabla\Phi_1$), and from (9) we derive the integral

$$p_1 + \partial\Phi_1/\partial t + \nabla\Phi_0\nabla\Phi_1 = 0. \tag{14}$$

Eliminating p_1 from (12) and (14) and assuming k = 1 in (13) we obtain the problem of determining the external flow in D to a first approximation

$$\begin{gathered}\Delta\Phi_1 = 0,\\ \partial\zeta_1/\partial t + \partial_1\partial\zeta_1/\partial\varphi - a_1\zeta_1 = v_{\rho 1}\ (\rho = 0),\\ \partial\Phi_1/\partial t + \nabla\Phi_0\nabla\Phi_1 - \zeta_1\partial p_0/\partial\rho = 0\ (\rho = 0),\\ \partial\Phi_1/\partial y|_{y=0} = W_1|_{s=\infty},\ \nabla\Phi_1 = \zeta_1 = 0 (t = 0).\end{gathered} \tag{15}$$

Thus, for asymptotic integration of system (1)-(6) we first solve the problem of the flow of an ideal liquid (8) and then determine the flow in the boundary layer close to the solid wall (10) and the first approximation of the external flow (15), and then the flow in the boundary layer close to the free boundary. Further, the higher approximations are determined in the same sequence.

Example. Consider the effect of low viscosity on the nonsteady state flow of an incompressible liquid in a circular cylinder. The region D_0 filled with ideal liquid represents the cylinder $-H(t) \leqslant z \leqslant H(t)$, $r \leqslant R(t)$. Here (r, θ, z) are cylindrical coordinates. The side boundary Γ_0 (r = R/t) is free, and on Γ_0 the pressure suffers a discontinuity ($p - p_0 = \sigma/R$, where σ is the surface tension). The impenetrable walls $z = \pm H(t)$ move opposite to one another with constant velocity V. The solution of Euler's equations ignoring gravitational forces has the form [4]

$$v_{r0} = \tau r,\ v_{z0} = -2\tau z,\ v_{\theta 0} = 0,\ p_0 = 0{,}5(\tau^2 + \tau_t)(R^2 - r^2) + \sigma/R,$$

$$\tau(t) = (\lambda/2)(1 - \lambda t)^{-1/2},\ H(t) = h(1 - \lambda t),\ R = R_0(1 - \lambda t)^{-1/2},\ \lambda = -V/h = \text{const}.$$

The free boundary Γ_0 with t = 0 is a circular cylinder and as t increases the cylinder becomes flattened to the plane z = 0.

Consideration of the viscosity leads to expansions (7) everywhere apart from a small region of the line contact of the free boundary and the walls S. The asymptotics of the flow in the neighborhood of the line of contact are constructed in [5]. The main term of expansion (7) w_0 satisfies equations of the form (10) in cylindrical coordinates. We will introduce the function ψ(s, t) using the equation $w_{r0} = r\partial\psi/\partial s$, $w_{z1} = 2\psi - v_{z1}|_{z=H}$, in which case, in the neighborhood of the wall z = H, Eq. (10) has the form

$$\begin{gathered}\Phi''' + 4\Phi\Phi'' - 2\Phi'^2 - 8\Phi' + 2s\Phi'' = 0,\\ \Phi'(0) = -1,\ \Phi(0) = \Phi'(\infty) = 0;\end{gathered} \tag{16}$$

here

$$s = \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda t}}\frac{H-z}{\varepsilon},\quad \Phi(s) = \frac{\psi\sqrt{1-\lambda t}}{\sqrt{\lambda}}.$$

Equation (16) was integrated numerically. The function Φ(s) decreases monotonically from zero to the minimum value γ= −0.2063 for s = ∞. Hence, we have determined $v_{z1}|_{z=\pm H} = \pm 2\gamma/\sqrt{1 - \lambda t}$.

We will now determine the first approximation v_1, p_1, and ζ_1 in the external region (outside D_S and D_Γ). In view of the potential nature of the flow of an ideal liquid we have

$$\partial\Phi_1/\partial t + \tau r\partial\Phi_1/\partial r - 2\tau z\partial\Phi_1/\partial z + p_1 = 0,$$

where $v_1 = \nabla\Phi_1$. For Φ_1, ζ_1 we obtain problem (15) in the region D_0

$$\Delta\Phi_1 = 0,$$
$$\partial\Phi_1/\partial t + \tau R\partial\Phi_1/\partial r - 2\tau z\partial\Phi_1/\partial z + (\tau^2 + \tau_t)R\zeta + \sigma(\partial^2\zeta_1/\partial z^2 - \zeta_1 R^{-2}) = 0 \ (r = R),$$
$$\partial\zeta_1/\partial t - 2\tau z\partial\zeta_1/\partial z - \tau\zeta_1 = \partial\Phi_1/\partial r \ (r = R(t)),$$
$$\frac{\partial\Phi_1}{\partial z} = \pm\frac{2\sqrt{\lambda}\gamma}{\sqrt{1-\lambda t}} \quad (z = \pm H).$$

Taking into account the symmetry of the flow the solution of the last problem can be obtained in the form

$$\Phi_1 = \frac{2\sqrt{\lambda}\gamma}{4h}(1-\lambda t)^{-3/2}(2z^2 - r^2) + \sum_{k=0}^{\infty}\Phi_{k1}(t) I_0\left(\frac{\pi k r}{h(1-\lambda t)}\right)\cos\frac{\pi k z}{1-\lambda t}, \quad \zeta_1 = \sum_{k=0}^{\infty}\zeta_{k1}(t)\cos\frac{\pi k z}{1-\lambda t}.$$

The functions Φ_{k1}, ζ_{k1} satisfy a system of ordinary differential equations and can be obtained numerically. In particular, ζ_{01}, Φ_{01} can be obtained explicitly:

$$\zeta_{01} = \frac{2\sqrt{\lambda}\gamma}{\lambda h}\frac{\sqrt{1-\lambda t} - 1}{1-\lambda t}.$$

The contribution to the elevation of the free boundary from the boundary layer functions h_k and q_k is of the second order of smallness and is ignored here. It follows from an analysis of the equation for $\zeta_1(t, z)$ that the free boundary is deformed with time, becoming more and more convex.

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